# Linear Buckling Analysis of Functionally Graded Plates Using Power-Law Function

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*Abstract*— **Functionally graded materials (FGM) are microscopically inhomogeneous materials with continuous and smoothly varying material composition and material properties across the thickness. They are widely used in high temperature environments such as nuclear reactors and rocket heat shields, as the development of localised high values of stress are eliminated due to the absence of discrete boundaries between constituent materials. In this paper, the buckling behaviour of FGM plate is modelled using spline finite strip method. The material property is idealised by power law function and classical plate theory is used for analysis. The effect of aspect ratio, power law index and material composition on the critical buckling load and buckling coefficient are examined.**

***Keywords—functionally graded material; buckling; power law; spline finite strip, inverse iteration***

## I. INTRODUCTION

Functionally graded materials (FGM) are special class of composites of two or more constituent phases with continuous smoothly varying material composition and material properties across the thickness [9]. One of the major drawbacks of traditional composites is that localised high stresses are developed at discrete boundaries between constituent material phases when exposed to high-temperature environment, leading to de-bonding, loss of stiffness and ultimately to failure of the structural member [7]. This problem of localised high stresses and resulting loss of strength is eliminated in FGM as there are no discrete boundaries between constituent materials. This makes FGMs ideal for high-temperature applications such as nuclear reactors, space vehicles etc. [9].

FGMs can be manufactured by ion implantation, shot peening, thermal spraying, electrophoretic deposition or chemical vapour deposition and the material composition and grading can be precisely controlled to have desired properties [11]. For example, in metal – ceramic FGM, ceramic constituent provides high-temperature resistance and protects the metal from oxidation, while the ductile metal constituent prevents fracture caused by stresses due to high-temperature gradient in a very short distance.

In a number of structural applications, FGMs are used in the form of plates. In thin plate structural members, buckling strength is a critical consideration for design, together with flexural and axial strength. Therefore, determination of critical buckling load and buckling coefficient for various material combinations and boundary conditions is important for structural applications of FGM. A number of research works on buckling analysis of FGMs have already been carried out [1], [5], [6], [10], [12]-[16].

## II. ANALYSIS OF FUNCTIONALLY GRADED PLATES

### A. General

The geometry of an elastic functionally graded plate [4] is shown in Fig. 1. The X-Y plane defines the mid-plane of the plate, and the z-axis originating at the middle surface of the plate is in the thickness direction.

The classical plate theory (CPT) and higher order shear deformation theories used for the analysis of isotropic plates can be extended to the analysis of FGM plates [11]. In the present work, CPT is used for the analysis of FGM plates. In CPT, it is assumed that the transverse normals to the mid-plane of the un-deformed plate remain straight and normal after deformation. The deformation is entirely due to the bending and in-plane stretching and the effect of transverse stresses are ignored [17].

### B. Mathematical idealization

The heterogeneous material properties, varying smoothly across the thickness of the plate, can be idealised by some homogenisation schemes for analysis. The various homogenization schemes used for idealising the variation of Young’s modulus (E) across the thickness are power-law, sigmoid and exponential functions and these are designated as PFGM, SFGM and EFGM respectively [9]. In the present work, idealisation by power-law function is used. As the effect of Poisson’s ratio in the deflection of structural members is negligible when compared to that of Young’s modulus [4], Poisson’s ratio is considered as a constant in the analysis.

The power law function ($g$ ) used for idealising the variation of E across the thickness (h) is defined by the equation,

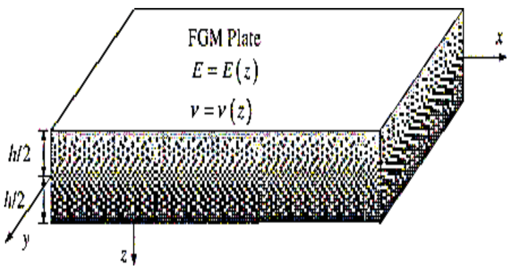


Fig. 1 The geometry of an FGM plate

$$g = \left(\frac{z + h/2}{h}\right)^p \tag{1}$$

where z is the distance from the mid surface in the z-direction.

Once the power function is obtained, the E can be obtained by the rule of mixtures as,

$$E = gE_1 + [1-g]E_2 \tag{2}$$

where $E_1$ and $E_2$ are the Young's moduli of the bottom and top surfaces (z = ±h/2) of the functionally graded plate respectively. The variation of E across the thickness of the plate as defined by the power law function [4] is as shown in Fig. 2.

### C. *Approximate methods of analysis*

Since exact analysis of buckling of FGM plates for complex boundary conditions and loading conditions is cumbersome and time consuming, approximate methods of analysis can be used. Finite element method (FEM), which is the most widely used approximate method of analysis for structural members, has a number of limitations when applied to stability and non-linear analysis of long thin rectangular plates, especially when iterative techniques are involved. In FEM, plate is discretised into a number of elements with approximately the same aspect ratio as that of the structural member, resulting in a large number of elements and consequently greater memory and computation time requirements. Finite strip method, which takes advantage of the prismatic nature of geometry by discretising the plate into strips instead of elements, can be effectively used to reduce the memory requirements without compromising on accuracy [8]. Here the displacement along the length of the strip is represented by a trial function and the displacement across the strip is represented by polynomial shape function.

In classical finite strip method (CFSM), trigonometric series is used as the trial function to represent the displacement along the strip. Since trigonometric displacement function is infinitely continuous, CFSM fails to deal with complex boundary conditions and partial and concentrated loads. Spline finite strip method (SFSM), which uses spline function to represent displacement in the longitudinal direction, can overcome the deficiencies of trigonometric function.

Spline function is a piecewise polynomial of $n^{th}$ degree, connected to the adjoining splines. Since the basic cubic spline

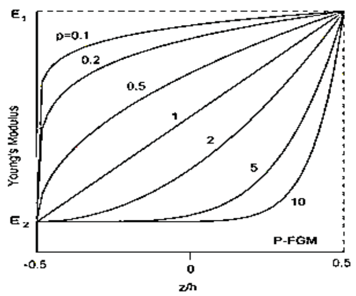


Fig. 2 Variation of E as per the power function

(B3 spline) is continuous over the first two derivatives and is discontinuous over the third derivative; it is suitable for representing plate-bending behaviour. B3 splines spanning over four consecutive sections are highly localised, resulting in highly banded stiffness matrix. In the present study, equally spaced cubic splines are used.

## III. Problem Formulation

### A. *Stiffness matrix formulation using CPT*

The stiffness matrix of thin functionally graded plates with thickness less than 0.1 times the lateral dimension can be formulated based on CPT. The normal and shear stresses $\sigma_x$ $\sigma_y$ and $\tau_{xy}$ are derived and the stress resultants are obtained by integrating the stresses over the thickness. The normal forces and bending moments are given by (3) and (4).

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} + \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{11} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{4}$$

The coefficients Aij, Bij and Cij are obtained by integrals as shown in (5).

$$\begin{aligned} A_{11} &= \int_{-h/2}^{h/2} \frac{E(z)}{1-\nu^2} dz & A_{12} &= \nu A_{11} & A_{66} &= \left(\frac{1-\nu}{2}\right) A_{11} \\ B_{11} &= \int_{-h/2}^{h/2} \frac{zE(z)}{1-\nu^2} dz & B_{12} &= \nu B_{11} & B_{66} &= \left(\frac{1-\nu}{2}\right) B_{11} \\ C_{11} &= \int_{-h/2}^{h/2} \frac{z^2 E(z)}{1-\nu^2} dz & C_{12} &= \nu C_{11} & C_{66} &= \left(\frac{1-\nu}{2}\right) C_{11} \end{aligned} \tag{5}$$

### B. *Eigenvalue buckling problem*

The load corresponding to elastic buckling is obtained by linear stability analysis, which is carried out by formulating the eigenvalue buckling problem and then solving it by any of the iterative methods. The equilibrium equation, neglecting the body forces, inertia effects and external lateral loading leads to the set of homogeneous equations,

$$[K]\{\delta\} - \lambda[K_G]\{\delta\} = 0 \tag{6}$$

where $[K_G]$ = Geometric stiffness matrix, and

$\lambda$ = Load parameter

Geometric stiffness matrix is the initial stress matrix, which depends only on the in-plane stresses and the geometric parameters. The smallest eigenvalue is the critical load and the eigenvector gives the buckling mode shapes [8]. There are a number of methods available for solution of eigenvalue problems. In this work, inverse iteration method is used for solving the eigenvalue buckling problem.

## IV. RESULTS AND DISCUSSION

A rectangular functionally graded plate of sides 100 mm and thickness 1 mm, simply supported on all four edges, acted upon by in-plane concentrated load is analysed. The variation of Young's modulus across the thickness of the plate is idealised by power law function and the Poisson's ratio (ν) is constant at 0.3. Only a quarter of the plate is analysed, taking advantage of the symmetry in geometry. The rectangular plate is discretised into four strips by five longitudinal nodal lines with five knots in each nodal line. The variations of displacements along the longitudinal and lateral directions are represented by a series of cubic splines and the polynomial shape function respectively. The critical buckling load and the buckling coefficient are obtained by spline finite strip method (SFSM), which is implemented in Visual C++. The influence of aspect ratio, power law index and the material composition ($E_1/E_2$ ratio) on the critical buckling load and buckling coefficient are studied.

### A. Validation of SFSM

For validating of the SFSM program implemented in Visual C++, the critical buckling load ($P_{cr}$) of isotropic square plate of side b and flexural stiffness D simply supported on four edges is obtained using the program. The buckling coefficient (k) is calculated by (7), and is compared with the exact value [3] as shown in Table 1.

$$k = \frac{P_{cr} b^2}{\pi^2 D} \tag{7}$$

Once SFSM program is validated, it can be extended to the analysis of FGM plates. Buckling coefficient of FGM plate can be calculated by (7), where D is replaced with $C_{11}$ obtained by (5).

### B. Effect of aspect ratio

Table 2 shows the critical buckling load and buckling coefficient of PFGM plates simply supported on four edges for various values of power-law index, $E_1/E_2$ ratio and aspect ratio. The variation of critical buckling load and buckling coefficient with aspect ratio for $E_1/E_2 = 2$ and p = 1 and p = 10 is shown in Fig. 3 and Fig. 4 respectively.

It is observed that square plate (aspect ratio = 1) has the minimum values of critical buckling load and buckling coefficient. The critical buckling load decreases sharply as the aspect ratio is increased from 0.5 to 1. As the aspect ratio is further increased, the critical buckling load rises steadily and reaches a maximum value at an aspect ratio of about 1.75. Up to an aspect ratio of 1.75, the plate buckles in the first mode and after this point, the plate buckles in the second mode. As the value of aspect ratio is further increased, the critical buckling load first decreases up to an aspect ratio of 3 and thereafter the critical buckling load again increases. After an aspect ratio of 4, the plate buckles in the third buckling mode.

### C. Effect of power law index

The critical buckling load and buckling coefficient of a square PFGM plate simply supported on four edges are calculated for various values of power-law index for $E_1/E_2 = 2$ and $E_1/E_2 = 5$. The results are shown in fig. 5 and fig. 6.

TABLE I. COMPARISON OF BUCKLING COEFFICIENT OF ISOTROPIC PLATE OBTAINED BY SFSM WITH EXACT VALUE

| **Young's modulus of elasticity** | **Buckling coefficient** | |
|---|---|---|
| | ***SFSM*** | ***Exact value*** |
| 2.1 X 105 | 4 | 4 |

TABLE II.VARIATION OF BUCKLING BEHAVIOUR WITH ASPECT RATIO

| **Aspect ratio** | **Critical buckling load, N** | | | | **Buckling coefficient** | | | |
|---|---|---|---|---|---|---|---|---|
| | ***$E_1/E_2 = 2$*** | | ***$E_1/E_2 = 5$*** | | ***$E_1/E_2 = 2$*** | | ***$E_1/E_2 = 5$*** | |
| | ***p =1*** | ***p = 10*** | ***p =1*** | ***p = 10*** | ***p =1*** | ***p = 10*** | ***p =1*** | ***p = 10*** |
| 0.5 | 86.117 | 70.118 | 62.038 | [illegible] | 6.05 | 6.18 | 5.45 | 5.64 |
| 1 | 55.638 | 45.026 | 41.359 | 25.173 | 3.91 | 3.97 | 3.63 | 3.72 |
| 2 | 65.628 | 53.125 | 48.851 | 29.730 | 4.61 | 4.68 | 4.29 | 4.39 |

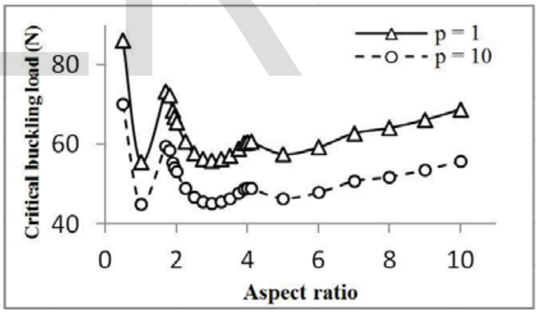


Fig. 3: Variation of critical buckling load with aspect ratio

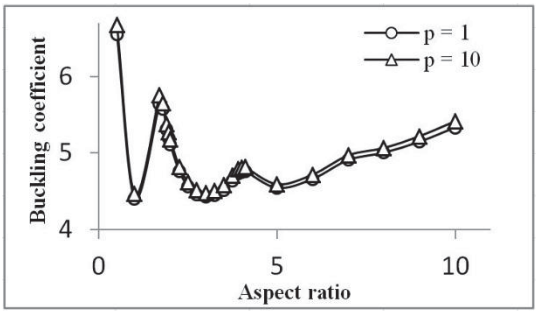


Fig. 4: Variation of buckling coefficient with aspect ratio

*D. Effect of power law index*

The critical buckling load and buckling coefficient of a square PFGM plate simply supported on four edges are calculated for various values of power-law index for $E_1/E_2$ = 2 and $E_1/E_2$ = 5. The results are shown in fig. 5 and fig. 6.

The critical buckling load decreases with increase in power law index (p). From Fig. 2 it is seen that for low values of p, the effect of the higher value of Young's modulus ($E_1$) is predominant over that of the lower value ($E_2$). Therefore, the stiffness of the plate at lower values of p is greater than that at higher values of p. This may be the reason for the reduction in critical buckling load with increase in p. The value of buckling coefficient decreases with increase in the value of p up to about 2 to 3 and thereafter it increases with the value of p.

*E. Effect of $E_1/E_2$ ratio*

The critical buckling load and buckling coefficient of a square PFGM plate simply supported on four edges are calculated for various values of $E_1/E_2$ ratio for p = 1 and p = 10. The results are shown in fig. 7 and fig. 8.

As the $E_1/E_2$ ratio increases, the critical buckling load and buckling coefficient of PFGM plate decrease. As the value of $E_1/E_2$ ratio increases, the flexural stiffness of the functionally graded plate decreases, as the value of $E_2$ is decreased keeping the value of $E_1$ constant. This is the reason for decrease in critical buckling load and buckling coefficient. At low values of $E_1/E_2$, the critical buckling load and buckling coefficient for various values of p are close together. With increase in the value of $E_1/E_2$, the divergence between the values of critical buckling loads for various values of p increases. But the values of buckling coefficients for various values of p are close together for all values of $E_1/E_2$ as shown in fig. 8.

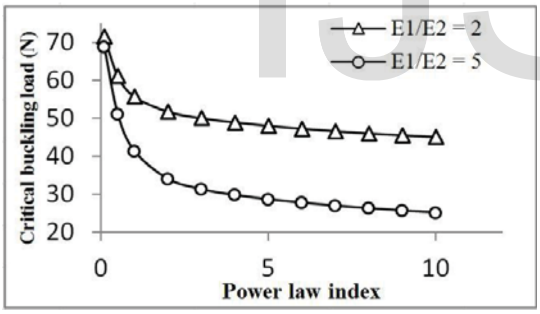


Fig. 5: Variation of critical buckling load with power law index

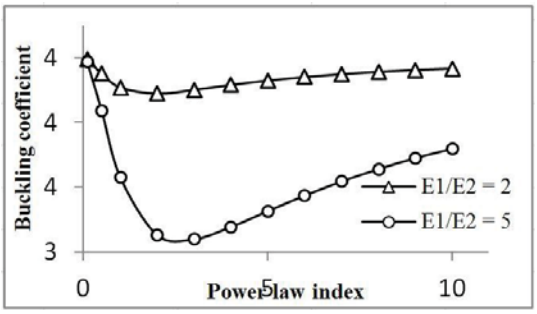


Fig. 6: Variation of buckling coefficient with power law index

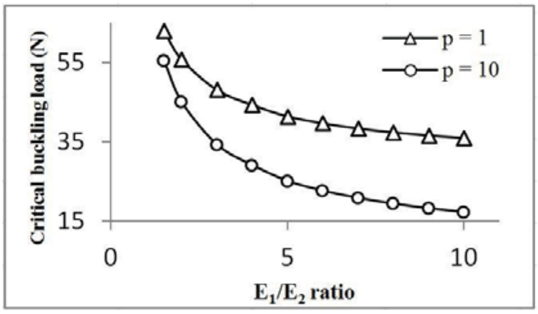


Fig. 7: Variation of critical buckling load with $E_1/E_2$ ratio

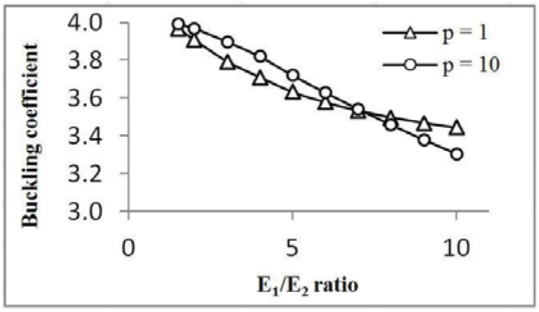


Fig. 8: Variation of buckling coefficient with $E_1/E_2$ ratio

## V. CONCLUSIONS

- The minimum value of critical buckling load and buckling coefficient for rectangular PFGM plate simply supported on all four edges occur for the square plate.
- The functionally graded plate simply supported on all the four edges buckle in the first mode up to an aspect ratio of 1.75, and thereafter the plate buckles in the second modes.
- The values of buckling coefficient of square plate are lower than that of the isotropic plate having Young's modulus equal to $E_1$.
- The critical buckling load of PFGM square plates decrease with increase in power law index, whereas the buckling coefficient first decreases and then increases.
- The critical buckling load and buckling coefficient of PFGM plate decrease with increase in $E_1/E_2$ ratio.
- The variation in the buckling behaviour of PFGM plates with $E_1/E_2$ ratio is very low at low values of p, but as the value of p is increased, the buckling behaviour shows wide variation with the change in value of p.

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